

1. Use the chain rule to prove that the derivative  
a) of an odd function is an even function

$f$  is even if  $f(-x) = f(x)$

$f$  is odd if  $f(-x) = -f(x)$

let  $f(x)$  ( $f(-x) = -f(x)$ ), be an odd function

Then we know that

$$f(-x) = -f(x)$$

Differentiate both sides with respect to  $x$

$$\frac{d[f(-x)]}{dx} = -f'(x)$$

Using chain rule in LHS, we can write

$$\frac{d[f(-x)]}{d(-x)} \cdot \frac{d(-x)}{dx} = -f'(x)$$

$$f'(-x) \cdot (-1) = -f'(x)$$

$$(-1) \cdot f'(-x) = -f'(x)$$

$$-f'(-x) = -f'(x)$$

Multiply both sides by  $-1$

$$f'(-x) = f'(x)$$

Hence proved that  $f'(x)$  is even.

b) Using the first principle (and a change of variables), show that the derivative of an even function is an odd function

$f$  is even:  $f(-x) = f(x)$

Example:  $f(x) = x^2$

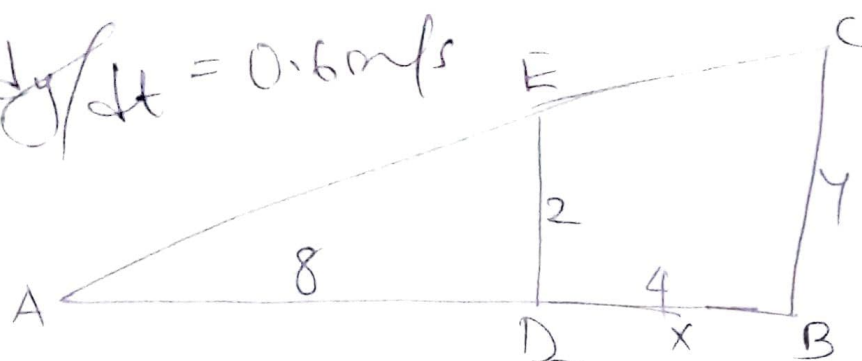
Differentiate both sides:

$$-f'(-x) = f'(x), \text{ chain rule,}$$

$$\text{or } f'(-x) = -f'(x)$$

Q.2.

$$\frac{dy}{dt} = 0.6 \text{ m/s}$$



The man is at D with head at E

Shadow on the wall is  $y = BC$

The two right angles  $\triangle ABC$  &  $\triangle ADE$  are similar, for angles, as such:

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{8}{12} = \frac{2}{y}, \therefore y = 3$$

① Distance of man from building =  $x$

" " Spotlight to the man =  $12 - x$

$$\frac{12 - x}{12} = \frac{2}{y} \quad 1 - \frac{1}{12}x = 2 \cdot \frac{1}{y}$$

derivative.

$$-\frac{1}{12} \frac{dx}{dt} = \frac{2}{y^2} \frac{dy}{dt}$$

$$-\frac{1}{12} dx = -2 \frac{1}{y^2} dy$$

divides both sides by  $dt$

$$-\frac{1}{12} \cdot \frac{dx}{dt} = -\frac{2}{y^2} \cdot \frac{dy}{dt}$$

At the specified moment

$$\frac{dx}{dt} = 1.6 \text{ m/s}$$

$y = 3$  let us plug in:

$$-\frac{1}{12} (1.6) = -\frac{2}{y^2} \cdot \frac{dy}{dt}$$

$$dy/dt = 1.6/12 / 2/9 = \frac{1.6}{12} \cdot \frac{9}{2} = 0.6 \text{ m/s}$$

3. Point A, Point B

Curve  $x^{2/3} + y^{2/3} = 9$ , tangent gradient  $= \sqrt{5/2}$

$$\sqrt[3]{y^{2/3}} = \sqrt[3]{9 - x^{2/3}}$$

$$y = \sqrt[3]{9 - x^{2/3}}$$

$$dy/dx = (9 - x^{2/3})^{1/2}$$

$$d/dx [x^n] = n x^{n-1}$$

$$= \frac{3}{2} (9 - x^{2/3})^{1/2-1}$$

$$\frac{3}{2} (9 - x^{2/3})^{1/2-1}$$

$$= \frac{3}{2} (9 - x^{2/3})^{1/2}$$

$$\frac{3}{2} (9 - x^{2/3})^{1/2} = \sqrt{5/2} \times \frac{2}{3}$$

$$(9 - x^{2/3})^{1/2} = \sqrt{5/3}$$

$$(\sqrt{9 - x^{2/3}})^2 = (\sqrt{5/3})^2$$

$$9 - x^{2/3} = \frac{5}{3}$$

$$-x^{2/3} = \frac{5}{3} - 9$$

$$x^{2/3} = 9 - \frac{5}{3}$$

$$\sqrt[3]{x^{2/3}} = \sqrt[3]{76/9}$$

$$x = \sqrt[3]{76/9}$$

$$76/9 + y^{2/3} = 9$$

$$y^{2/3} = 9 - 76/9$$

$$(\sqrt[3]{76/9}, \sqrt[3]{5/3})$$

$$= \frac{5}{9}$$

$$y = \sqrt[3]{5/9}$$

Q4. Show that the following limit exists

$$\lim_{x \rightarrow c} (\ln x)^{\frac{1}{x-c}} \quad \ln(a^b) = b \ln(a)$$

$$e^{\ln(a)} = a$$

$$= \lim_{x \rightarrow c} (\ln x)^{\frac{1}{x-c}} = \lim_{x \rightarrow c} e^{\ln((\ln x)^{\frac{1}{x-c}})}$$

$$\lim_{x \rightarrow c} (\ln x)^{\frac{1}{x-c}} = \lim_{x \rightarrow c} e^{\ln((\ln x)^{\frac{1}{x-c}})}$$

$$\lim_{x \rightarrow c} e^{\ln((\ln x)^{\frac{1}{x-c}})} = e^{\lim_{x \rightarrow c} \frac{1}{x-c} \cdot \ln(\ln x)} =$$

$$e^{\lim_{x \rightarrow c} \frac{\ln(\ln x)}{x-c}}$$

$$\lim_{x \rightarrow c} e^{\ln((\ln x)^{\frac{1}{x-c}})} = e^{\lim_{x \rightarrow c} \frac{\ln(\ln x)}{x-c}}$$

$$(L'Hopital's rule) = e^{\lim_{x \rightarrow c} \frac{1}{\ln x} \cdot \frac{1}{x-c}}$$

$$= e^0 = 1$$

Q5. Let  $a, b \in \mathbb{R}$  be constants

$$\lim_{x \rightarrow 1} \frac{ax^4 + bx^3 + 1}{(x-1)\sin(\pi x)}$$

$$\lim_{x \rightarrow 1} \frac{a1^4 + b1^3 + 1}{(1-1)\sin(\pi \cdot 1)} = \frac{a1^4 + b1^3 + 1}{0} = -\infty$$

As  $x$  approaches 1, if  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = L$  then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$

$$\lim_{x \rightarrow 1} \frac{ax^4 + bx^3 + 1}{(x-1)\sin(\pi x)}$$

$$\text{where } x = 0.999 = \frac{a \cdot 0.999^4 + b \cdot 0.999^3 + 1}{(0.999 - 1)\sin(\pi \cdot 0.999)}$$

(5)

Simplify

$$\frac{9.0999^4 + 6.0.999^3 + 1}{(0.999-1) \sin(\pi.0999)}$$

$$= \frac{0.99600...9 + 0.997002996 + 1}{3.14159...E-6}$$

at  $\lim$ 

$$x \rightarrow 0.9$$

$$= \frac{9.0.9^4 + 6.0.9^3 + 1}{(0.9-1) \sin(\pi 0.9)}$$

$$= \frac{0.6561 a + 0.729 b + 1}{0.03090...}$$

Ex. 6. Mean Value Theorem to  $f(x) = \sqrt{x}$   
 Interval  $[100, 102]$  prove  
 $\frac{1}{11} < \sqrt{102} - 10 < \frac{1}{10}$

$f(x) = \sqrt{x}$  is continuous over  $[100, 102]$   
 and differentiable over  $(100, 102)$

Derivative

$$f'(x) = \frac{1}{2\sqrt{x}} \quad f'(c) = \frac{1}{2\sqrt{c}}$$

~~The slope of the line is~~

$$\frac{f(102) - f(100)}{102 - 100} = \frac{\sqrt{102} - \sqrt{100}}{102 - 100} = \frac{\sqrt{102} - 10}{2}$$

$$f'(c) = \frac{\sqrt{102} - 10}{2} \quad c \in (100, 102)$$

$$2 \cdot \frac{1}{2\sqrt{c}} = \frac{\sqrt{102} - 10}{2} \times 2$$

$$\sqrt{102} - 10 < \frac{1}{10}$$

$$\frac{1}{\sqrt{c}} = \sqrt{102} - 10$$

$$\sqrt{c} = \frac{1}{\sqrt{102} - 10}$$

$$c = \left( \frac{1}{\sqrt{102} - 10} \right)^2$$

$$\frac{1}{\sqrt{102+2}} < \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{102-2}}$$

$$\frac{1}{\sqrt{122}} < \sqrt{102} - 10 < \frac{1}{\sqrt{100}}$$

$$\frac{1}{11} < \sqrt{102} - 10 < \frac{1}{10}$$

7. Curve  $4y = x^2$  To get gradient  
 $y = x^2/4$  (A)

Curve  $x = 2y^2$  (B)  
 $y = \sqrt{x/2}$

$$\frac{dy}{dx} \left( \frac{x^2}{4} \right) = \frac{x}{2}$$

$$\frac{dy}{dx} \left( \sqrt{x/2} \right) = \frac{1}{2\sqrt{2}\sqrt{x}}$$

Since gradient of the line is the same

$$\cancel{\frac{x}{2}} = \frac{1}{2\sqrt{2}\sqrt{x}} \times 2$$

$$\sqrt{x} \cdot x = \frac{1}{\sqrt{2}\sqrt{x}}$$

$$(\sqrt{x})^2 \cdot (x) = \left( \frac{1}{\sqrt{2}} \right)^2$$

$$x \cdot x^2 = \frac{1}{2}$$

$$\sqrt[3]{x^3} = \sqrt[3]{\frac{1}{2}}$$

$$x = \sqrt[3]{\frac{1}{2}}$$

$$y = (3\sqrt[3]{1/2})/4$$

$$y = \sqrt{3\sqrt[3]{1/2}/2}$$

$$A (0.7937, 0.1984)$$

$$B (0.7937, 0.62996)$$

$$d(AB) = \sqrt{[(x_1 - y_1)^2 + (x_2 - y_2)^2]}$$

$$= \sqrt{(0.5953)^2 + (0.16374)^2}$$

$$0.3544 + 0.0268$$

$$= \sqrt{0.3812}$$

$$= \underline{0.6174 \text{ units}}$$

$$3 \text{ cm}^2 / 32 \text{ cm}^2$$

$$3 \text{ cm}^2 \text{ top + bottom}$$

$$2 \text{ cm}^2 \text{ curved}$$

$$V = \pi r^2 h$$

$$S.A = 2\pi r^2 + 2\pi r h$$

$$\text{now } h = \frac{V}{\pi r^2}$$

$$S.A = 2\pi r^2 + 2\pi r \left( \frac{V}{\pi r^2} \right)$$

$$= 2\pi r^2 + 2 \frac{V}{r}$$

$$\frac{d}{dr} (2\pi r^2 + 2 \frac{V}{r}) = 0$$

$$4\pi r - \frac{2V}{r^2} = 0$$

$$4\pi r^3 = 2V$$

$$r^3 = \frac{V}{2\pi}$$

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

Since: top + bottom cost  $3 \text{ cm}^2$  &  $2 \text{ cm}^2$  curved

$$\text{cost: } 3(2\pi r^2) + 2(2\pi r h)$$

$$6\pi r^2 + 4\pi r h$$

$$6\pi r^2 = -4\pi \frac{V}{r^2}$$

$$6r^4 = -4V$$

Since top + bottom

cost is  $3 \text{ cm}^2$

& curved is  $2 \text{ cm}^2$

$$3(2\pi r^2) + 2(2\pi r h)$$

$$6\pi r^2 + 4\pi r \left( \frac{V}{\pi r^2} \right)$$

$$6\pi r^2 + 4 \frac{V}{r}$$

$$r \cdot 6\pi r^2 = 4 \frac{V}{r} \times r$$

$$6\pi r^3 = 4V$$

$$6\pi r^3 = 4V$$

$$r^3 = \frac{4V}{6\pi}$$

$$r^3 = \frac{2V}{3\pi}$$

$$r = \sqrt[3]{\frac{2V}{3\pi}}$$

9.  $f(x) = xe^{-\frac{x^2}{2}}$

(a)  $f(-x) = -xe^{-\frac{(-x)^2}{2}}$   
 $= -xe^{-\frac{x^2}{2}}$   
 $= -f(x)$

$f(x) \neq f(-x)$

Thus  $f(x)$  is odd

(b) horizontal and vertical asymptotes of  $f(x)$

Vertical asymptotes of  $f$   
 None

Horizontal asymptotes of  $f$

$y = 0$

(c) Critical points and intervals of increase and decrease of the function.

$xe^{-\frac{x^2}{2}}$  Where  $f'(x)$  is equal to zero or undefined

$x = -1, x = 1$

(d) Monotone intervals of  $xe^{-\frac{x^2}{2}}$

Decreasing:  $-\infty < x < -1$

Increasing:  $-1 < x < 1$

Decreasing:  $1 < x < \infty$

(e) Inflection points and intervals of concavity of  $f$

Point on graph with second derivative

if  $f''(x) > 0$  then concave upwards

$f''(x) < 0$  " " downwards.

Where  $f''$  is equal to zero or undefined.

$(x = -3^{\frac{1}{2}}, x = 0, x = 3^{\frac{1}{2}})$   
 $(0, 0), (3^{\frac{1}{2}}, (\frac{3}{e^3})^{\frac{1}{2}})$

$$\left(-3^{1/2}, -\left(\frac{3}{e^3}\right)^{1/2}\right), (0,0), \left(3^{1/2}, \left(\frac{3}{e^3}\right)^{1/2}\right)$$